

The modification of the energy spectrum of charged particles by exotic open 4-smoothness via superstring theory

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In this paper we present a model where the modified Landau-like levels of charged particles in a magnetic field are determined due to the modified smoothness of \mathbb{R}^4 as underlying structure of the Minkowski spacetime. Then the standard smoothness of \mathbb{R}^4 is shifted to the exotic \mathbb{R}_k^4 , $k = 2p$, $p = 1, 2, \dots$. This is achieved by superstring theory using gravitational backreaction induced from a strong, almost constant magnetic field on standard \mathbb{R}^4 . The exact string background containing flat \mathbb{R}^4 is replaced consistently by the curved geometry of $SU(2)_k \times \mathbb{R}$ as part of the modified exact backgrounds. This corresponds to the change of smoothness on \mathbb{R}^4 from the standard \mathbb{R}^4 to some exotic \mathbb{R}_k^4 . The calculations of the spectra are using the CFT marginal deformations and Wess-Zumino-Witten (WZW) models. The marginal deformations capture the effects of the magnetic field as well as its gravitational backreactions. The spectra depend on even level k of WZW on $SU(2)$. At the same time the WZ term as element of $H^3(SU(2), \mathbb{R})$ determines also the exotic smooth \mathbb{R}_k^4 . As the consequence we obtain a non-zero mass-gap emerges in the spectrum induced from the presence of an exotic \mathbb{R}_k^4 .

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I. INTRODUCTION

For every charged quantum particle, say e , moving through a non-flat gravitational background there should

exist a high energy limit where one has to include quantum gravitational effects. Additionally this particle cannot be described by perturbative field theory any longer. In dimension 4 quantum gravity (QG) is still far from being a complete theory. The classical part is well understood by using (pseudo-)Riemannian smooth geometry but the transformation of classical (geometric) gravity to QG is a mystery especially from the purely mathematical point of view. In this paper we address both issues and show how this transformation can be done by a (special) smooth Riemannian 4-geometry.

QG is already defined in 10 spacetime dimensions as superstring theory (though not as any complete quantum field theory). There exist many techniques worked out in superstring theory which refer to dimension 4 like: compactification, flux stabilization, brane configuration model-buildings, brane worlds, holography or AdS/CFT and some others. But the transition from higher dimensional string theory to 4-dimensional physics using these methods is rather ambiguous. Recently two of us have proposed [6, 8, 10, 11] to incorporate 4-dimensional effects into string theory via its connections with an inherently 4-dimensional differential geometric phenomenon: exotic smoothness of topologically trivial \mathbb{R}^4 . Among all \mathbb{R}^n only the case $n = 4$ allows for different smoothings of the Euclidean space, i.e. an atlas (or a system of reference frames) non-diffeomorphic to the usual global coordinate patch of the \mathbb{R}^n (one chart). Especially we proposed how to transfer some string theory techniques for backgrounds containing an exotic \mathbb{R}^4 as 4-dimensional part. String theory for these backgrounds define in a new way the connections with 4-dimensional physics. This new 4-dimensional window of superstring theory refers as a tool to exact backgrounds in any order of α' . Such backgrounds are rather exceptional, though very desirable, in superstring theory (see e.g. [23, 24]). We will

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make extensive use of these backgrounds also in this paper and show direct applicability of the link between *strings* and *4-exoticness* in deriving physical results in dimension 4. Especially we will answer the questions: Given exotic smooth differential structure on \mathbb{R}^4 , i) can one describe physical effects due to the propagation of strong magnetic field on this exotic 4-geometry underlying Minkowski spacetime? and ii) How can one deal consistently with gravity of non-flat exotic metrics on \mathbb{R}^4 at the quantum level? The answers depend strongly on the QG regime. In particular the QG backreaction of extremely strong magnetic fields H is evaluated. We obtain the energy spectra of spin and spinless charged particles in a magnetic field on these exotic backgrounds by superstring theory. The geometry used in these calculations is induced by some exotic \mathbb{R}^4 carrying both gravity regimes, classical gravity on a non-flat 4-manifold [26], and the QG regime [6, 8].

In superstring theory one can consistently grasp the QG effects by changing the exact background containing the flat \mathbb{R}^4 to a background with a curved 4-dimensional part, i.e. $\mathbb{R}^4 \times K^6 \rightarrow SU(2)_k \times \mathbb{R}_\phi \times K^6$. The $SU(2)_k \times \mathbb{R}_\phi$ part is the background of the linear dilaton $\phi \sim QX^0$ in the superconformal theory on $SU(2) \times \mathbb{R}$ [8, 12, 18] where Q is the charge. This change of the backgrounds allows us to derive 4D effects from QG backreaction of strong (magnetic) fields. These effects can be calculated by using methods from 2D CFT and $SU(2)_k$ WZW model [16, 18, 24]. Then the change of the backgrounds corresponds to the change of smoothness on 4-space: $\mathbb{R}^4 \rightarrow \mathbb{R}_k^4$ as global structure of the local Minkowskian spacetime.

In our paper, we consider some energy spectra of particles on S^3 with the presence of a magnetic field (Subsec. IV.1). The spectra are modified by the consideration of a sufficiently strong magnetic field leading also to quantum gravity contributions. The exact solutions can be derived from string theory. Then adding a magnetic field to the string theory backgrounds containing a smooth flat \mathbb{R}^4 results in some curved background by replacing the flat \mathbb{R}^4 (Subsec. III.1). In closed string theory, one has to include the effect of the gravitational field since even a constant magnetic field carries energy and spacetime has to be curved. Thus the flat Euclidean 4-space is replaced by the curved $S^3 \times \mathbb{R}$ geometry. This replacement is rather universal (i.e. independent on the type of string theory) and results in the appearance of the level k WZW on $SU(2)$ [18].

From the geometrical point of view the integral class $k \in H^3(S^3, \mathbb{Z})$ can be realized by some exotic smooth \mathbb{R}_k^4 where the effects of the exotic smoothness can be localized on some $S^3 \subset \mathbb{R}^4$. This is a kind of a geometric *localization principle* as discussed in [5]. Thus, effects of certain exotic \mathbb{R}_k^4 's, when localized on S^3 , give rise to the level k WZW on $SU(2)$. The reason for this coincidence is deeply rooted in the Godbillon-Vey (GV) class of some codimension one foliations of S^3 . Its evaluation on the fundamental class of S^3 is precisely the Wess-Zumino (WZ) term of the WZW model. Furthermore different GV classes of the foliation correspond to different exotic

\mathbb{R}_k^4 (with respect to some parameter family of exotic \mathbb{R}^4 's, the so-called radial family). This is discussed in the next Section.

The effects of strong magnetic fields and its corresponding gravity backreactions are obtained as suitable marginal deformations of the WZW model on $SU(2)$ (Subsec. III.1 and App. A). The string spectra in the background $SU(2)_k \times \mathbb{R}$ depend on these deformations. In Subsec. IV we relate these spectra with effects caused by exotic \mathbb{R}_k^4 's. Especially we discuss the large volume limit $k \rightarrow \infty$ corresponding to the flat \mathbb{R}^4 and the rescaling of the H field. Then flat string theory spectra deformed by H including gravitational effects do not depend on k . However, the exchange of the flat standard \mathbb{R}^4 by a non-flat exotic \mathbb{R}_k^4 gives a finite k in the WZW model. Then the dependence of the spectra (see Eqs. (27), (24)) on the level k reflects precisely the effects of the exotic \mathbb{R}_k^4 .

The whole analysis in this paper used deep mathematical structures to derive the physical results from some exotic \mathbb{R}_k^4 . As we mentioned above, exotic \mathbb{R}_k^4 are non-flat Riemannian manifolds, i.e. non-trivial solutions of Einstein's equations [26]. As shown here, this effect remains true for the regime where quantum corrections to gravity become important. Here we found that the structure of exotic \mathbb{R}_k^4 and its connection to string backgrounds are the appropriate tools for the calculation of the corrections.

II. SMALL EXOTIC \mathbb{R}_k^4 , FOLIATIONS AND WEISS-ZUMINO TERM

An exotic \mathbb{R}^4 is a topological space with \mathbb{R}^4 -topology but with a different (i.e. non-diffeomorphic) smoothness structure than the standard \mathbb{R}_{std}^4 getting its differential structure from the product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The exotic \mathbb{R}^4 is the only Euclidean space \mathbb{R}^n with an exotic smoothness structure. The exotic \mathbb{R}^4 can be constructed in two ways: by the failure to arbitrarily split a smooth 4-manifold into pieces (large exotic \mathbb{R}^4) and by the failure of the so-called smooth h-cobordism theorem (small exotic \mathbb{R}^4). Here we deal with the class of small exotic \mathbb{R}^4 's which allowing the smooth embedding of a 3-sphere. In [13], a family of uncountable many different small exotic \mathbb{R}^4 was constructed, the radial family depending on one parameter, the radius ρ . In the following we will use a countable subfamily \mathbb{R}_k^4 of this family as explained below.

There are widely discussed difficulties with explicit coordinate descriptions (see e.g. [3]) of different differential structures on \mathbb{R}^4 (and on other open 4-manifolds). In a series of recent papers, we were however able to relate these 4-exotics with some other structures on S^3 (see e.g. [4, 5, 7]). This 3-sphere S^3 is supposed to fulfill specific topological conditions: it has to lie in the ambient \mathbb{R}^4 as part of the boundary of some compact 4-submanifold (the so-called Akbulut cork) with boundary. If all properties are met, one can prove that exotic smoothness of the \mathbb{R}^4 is tightly related with codimension-one foliations

of this 3-sphere S^3 , hence with the 3-rd real cohomology classes of S^3 [5]:

The exotic \mathbb{R}^4 's from the radial family are determined by codimension-1 foliations \mathcal{F} 's with non-vanishing Godbillon-Vey (GV) class $GV(\mathcal{F})$ in $H^3(S^3, \mathbb{R})$ of a 3-sphere lying at the boundary of the Akbulut corks of \mathbb{R}^4 's. The radius ρ in the family and the Godbillon-Vey number GV as pairing $\langle GV(\mathcal{F}), [S^3] \rangle$ between the GV class and the fundamental class $[S^3] \in H_3(S^3)$ are related by $GV = \rho^2$. We say: the exoticness is localized at a 3-sphere inside the small exotic \mathbb{R}^4 (seen as a submanifold of \mathbb{R}^4).

In the particular case of the integral class as element of $H^3(S^3, \mathbb{Z})$ one obtains the relation between the exotic \mathbb{R}_k^4 , $k \in \mathbb{Z}$ and the WZ term of the k WZW model on $SU(2)$. In [5], we considered particularly this case. Topologically, this case refers to flat $PSL(2, \mathbb{R})$ -bundles over the space $(S^2 \setminus \{k \text{ punctures}\}) \times S^1$ where the gluing of k solid tori produces a 3-sphere (so-called Heegard decomposition). Then one obtains the relation [5]:

$$\frac{1}{(4\pi)^2} \langle GV(\mathcal{F}), [S^3] \rangle = \frac{1}{(4\pi)^2} \int_{S^3} GV(\mathcal{F}) = \pm(2-k) \quad (1)$$

in dependence on the orientation of the fundamental class $[S^3]$. Now we interpret the Godbillon-Vey invariant as WZ term. For that purpose we use the group structure $SU(2) = S^3$ of the 3-sphere S^3 and identify $SU(2) = S^3$. Let $g \in SU(2)$ be a unitary matrix with $\det g = -1$. The left invariant 1-form $g^{-1}dg$ generates locally the cotangent space connected to the unit. The forms $\omega_k = Tr((g^{-1}dg)^k)$ are complex k -forms generating the deRham cohomology of the Lie group. The cohomology classes of the forms ω_1, ω_2 vanish and $\omega_3 \in H^3(SU(2), \mathbb{R})$ generates the cohomology group. Then we obtain for the integral of the generator

$$\frac{1}{8\pi^2} \int_{S^3=SU(2)} \omega_3 = 1 \quad .$$

This integral can be interpreted as winding number of g . Now we consider a smooth map $G : S^3 \rightarrow SU(2)$ with 3-form $\Omega_3 = Tr((G^{-1}dG)^3)$ so that the integral

$$\frac{1}{8\pi^2} \int_{S^3=SU(2)} \Omega_3 = \frac{1}{8\pi^2} \int_{S^3} Tr((G^{-1}dG)^3) \in \mathbb{Z}$$

is the winding number of G . Every Godbillon-Vey class with integer value like (1) is generated by a 3-form Ω_3 . Therefore the Godbillon-Vey class is the WZ term of the $SU(2)_k$. Thus we obtain the relation:

The structure of exotic \mathbb{R}_k^4 's, $k \in \mathbb{Z}$ from the radial family determines the WZ term of the k WZW model on $SU(2)$.

Originally this WZ term was introduced to cancel the conformal anomaly of the classical σ -model on $SU(2)$. Thus we have a way to include this cancellation term by using smooth 4-geometry. Then we can argue that the

smoothness of the embedding space \mathbb{R}^4 of the 3-sphere S^3 depends on its codimension-1 foliation. In the exotic case \mathbb{R}_k^4 one obtains the WZ term of the classical σ -model with target $S^3 = SU(2)$. Furthermore we have the important correlation:

The change of smoothness from an exotic \mathbb{R}_k^4 to an exotic \mathbb{R}_l^4 , $k, l \in \mathbb{Z}$ both from the radial family, corresponds to the change of the level from k to l of the WZW model on $SU(2)$, i.e. $k \text{ WZW} \rightarrow l \text{ WZW}$.

Let us consider now the end of the exotic \mathbb{R}_k^4 i.e. $S^3 \times \mathbb{R}$. This end cannot be standard smooth [15] and it is in fact an exotic smooth $S^3 \times_{\Theta_k} \mathbb{R}$ [14]. Using the connection of the exotic \mathbb{R}_k^4 with the WZ term above, we have determined the „quantized” geometry of $SU(2)_k \times \mathbb{R}$ by relating it to the exotic geometry of the end of \mathbb{R}_k^4 . Especially the appearance of the $SU(2)_k \times \mathbb{R}$ is a source for various further constructions. In particular, the gravitational effects of \mathbb{R}_k^4 on the quantum level are determined via string theory by replacing consistently the flat \mathbb{R}^4 part of the background by the curved 4-space $SU(2)_k \times \mathbb{R}$.

III. SUPERSTRING EXACT SOLUTIONS AND EXOTIC 4-GEOMETRIES

We want to use the relation from the previous section and the results of [6, 8] to evaluate some physical effects from the exotic \mathbb{R}_k^4 in the QG regime. In the following we simply suppose that there is a locally Minkowskian smooth metric on \mathbb{R}_k^4 .

Exotic \mathbb{R}_k^4 is a smooth Riemannian manifold but its structure essentially deals with non-commutative geometry and quantization [7]. The connection with string exact backgrounds was also recognized [6, 8]. Especially with the topological assumptions above (see Sec. II) we obtained the following correlation in our previous work:

The change of the smoothness from the standard \mathbb{R}^4 to exotic \mathbb{R}_k^4 , corresponds to the change of exact string backgrounds from $\mathbb{R}^4 \times K^6$ to $SU(2)_k \times \mathbb{R}_\phi \times K^6$.

Let us note the importance of the exotic smoothness structure. Otherwise we are left with separate regimes of smooth 4-geometry (GR) and superstring theory (QG).

In string theory the above change of backgrounds is one way to include effects of a strong magnetic field H and its gravitational backreaction in the 4D part of the background [18]. These results show the existence of a consistent, from the point of view of QG and field theory, way to change the geometry of flat \mathbb{R}^4 to the geometry of a non-flat $SU(2)_k \times \mathbb{R}$. The calculations of correlations functions are performed in heterotic and type II superstring theories along with superconformal world-sheet symmetry where one consistently changes between backgrounds (see above). The coincidence of the 4D parts of the backgrounds with structures derived from small exotic \mathbb{R}_k^4 is a tool for the calculation of strong H QG effects, when H is defined on exotic \mathbb{R}_k^4 . The detailed description of the exact supersymmetric $\mathcal{N} = 4$, $c = 4$

backgrounds (as representations of the superconformal algebra) can be found in [1, 18]. In the next section following [18] we discuss the deformations of these backgrounds from the point of view of σ -model and CFT constructions (see Appendix A) which represent the effects of the strong magnetic field H and gravity backreactions on the backgrounds.

III.1. Marginally deformed exact string backgrounds

For a given exotic \mathbb{R}_k^4 , $k = 1, 2, \dots$, from the radial family [5, 8] one can consider the exact background of closed string theory $\mathbb{R}_\phi \times SU(2)_k \times \mathbb{R}^{5,1}$ [8] where the part of the background $\mathbb{R}_\phi \times SU(2)_k$ appears only when the flat standard \mathbb{R}^4 is replaced by the exotic \mathbb{R}_k^4 . As shown in

[18] the replacement of a flat \mathbb{R}^4 by $\mathbb{R}_\phi \times SU(2)_k$ in the string background $\mathbb{R}^4 \times K^6$ is a tool to include strong (almost) constant magnetic field effects in 4-dimensions and its corresponding quantum gravitational backreactions. This replacement was used to calculate the 4-dimensional magneto-gravitational deformations of the spectra of charged particles [18]. We remark that the change from the flat \mathbb{R}^4 to the curved 4-dimensional part: $\mathbb{R}^4 \times K^6 \rightarrow SU(2)_k \times \mathbb{R}_\phi \times K^6$ is performed certainly under the presence of supersymmetry in 10 dimensions. Here, we consider the presence of supersymmetry as a technical tool to effectively perform QG calculations. Firstly, let us recall following [18] how the exact backgrounds and their deformations appear on the level of σ -models. We consider the $SO(3)_{k/2} \times \mathbb{R}_Q$ CFT case which is the result of a projective map $SU(2)_k \times \mathbb{R}_Q \rightarrow SO(3)_{k/2} \times \mathbb{R}_Q$ for k even. In this case, the action for heterotic σ -model is given by:

$$S_4 = \frac{k}{4} I_{SO(3)}(\alpha, \beta, \gamma) + \frac{1}{2\pi} \int d^2 z \left[\partial x^0 \bar{\partial} x^0 + \psi^0 \partial \psi^0 + \sum_{a=1}^3 \psi^a \partial \psi^a \right] + \frac{Q}{4\pi} \int \sqrt{g} R^{(2)} x^0 \quad (2)$$

where $I_{SO(3)}(\alpha, \beta, \gamma) = \frac{1}{2\pi} \int d^2 z [\partial \alpha \bar{\partial} \alpha + \partial \beta \bar{\partial} \beta + \partial \gamma \bar{\partial} \gamma + 2 \cos \beta \partial \alpha \bar{\partial} \gamma]$ in the Euler angles of $SU(2) = S^3$. Then the bosonic σ -model action reads in general:

$$S = \frac{1}{2\pi} \int d^2 z (G_{\mu\nu} + B_{\mu\nu}) \partial x^\mu \bar{\partial} x^\nu + \frac{1}{4\pi} \int \sqrt{g} R^{(2)} \Phi(x) \quad (3)$$

By comparison with (2) we obtain the non-zero background fields as:

$$G_{00} = 1, \quad G_{\alpha\alpha} = G_{\beta\beta} = G_{\gamma\gamma} = \frac{k}{4}, \quad G_{\alpha\gamma} = \frac{k}{4} \cos \beta \\ B_{\alpha\gamma} = \frac{k}{4} \cos \beta, \quad \Phi = Q x^0 = \frac{x^0}{\sqrt{k+2}}. \quad (4)$$

Following [25], one can decompose the supersymmetric WZW model into a bosonic $SU(2)_{k-2}$ part with affine currents J^i and into three free fermions ψ^a , $a = 1, 2, 3$ in the adjoint representation of $SU(2)$. Supersymmetry with $\mathcal{N} = 1$ implies for the affine currents the expression $\mathcal{J}^a = J^a - \frac{i}{2} \epsilon^{abc} \psi^b \psi^c$. After introducing the complex fermion combinations $\psi^\pm = \frac{1}{\sqrt{2}} (\psi^1 \pm i \psi^2)$ and the corresponding change of the affine bosonic currents $J^\pm = J^1 \pm i J^2$, the supersymmetric affine currents read:

$$\mathcal{J}^3 = J^3 + \psi^+ \psi^-, \quad \mathcal{J}^\pm = J^\pm \pm \sqrt{2} \psi^3 \psi^\pm \quad (5)$$

Let us redefine the indices in the fermion fields as: $+ \rightarrow 1$, $- \rightarrow 2$, then $\mathcal{J}^3 = J^3 + \psi^1 \psi^2$. From the σ -model point of view, the vertex for the magnetic field H on the 4-dimensional curved space $\mathbb{R}_\phi \times SU(2)_k$ is the exact

marginal operator given by $V_m = H(J^3 + \psi^1 \psi^2) \bar{\mathcal{J}}^a$. Similarly the vertex for the corresponding gravitational part is $V_{gr} = \mathcal{R}(J^3 + \psi^1 \psi^2) \bar{\mathcal{J}}^3$, and represents the really marginal deformations too. The shape of these operators follow from the fact that the marginal deformations of the WZW model can be in general constructed as bilinear expressions in the currents $J, \bar{\mathcal{J}}$ of the model [24]:

$$\mathcal{O}(z, \bar{z}) = \sum_{i,j} c_{ij} J^i(z) \bar{\mathcal{J}}^j(\bar{z}) \quad (6)$$

where $J^i, \bar{\mathcal{J}}^j$ are left and right-moving affine currents, respectively [24]. Let $\mathfrak{g}, \bar{\mathfrak{g}}$ be the holomorphic and antiholomorphic Kac-Moody affine algebras of the $SU(2)_k$ WZW model (seen as Lie algebra). The currents $J^i, \bar{\mathcal{J}}^j$ span the abelian Lie sub-algebras $\mathfrak{h}, \bar{\mathfrak{h}}$ of the Lie algebras $\mathfrak{g}, \bar{\mathfrak{g}}$. Then the Lie groups $U(1)^d, U(1)^{\bar{d}}$ with $d = \dim \mathfrak{h}$ and $\bar{d} = \dim \bar{\mathfrak{h}}$ are associated to the Lie algebras $\mathfrak{h}, \bar{\mathfrak{h}}$. The complete class of the marginal deformations of the WZW model (6) are determined by $\frac{O(d,d)}{O(d) \times O(d)}$ interpreted as transformations of the lattice of the charges ([24], p. 20). In case of the single $U(1)$ subalgebra (and a single deforming field) we have $O(1,1)$ deformations (see Appendix A).

Here, following [18], we consider a covariantly constant magnetic field $H_i^a = \epsilon^{ijk} F_{jk}^a$ and a constant curvature $\mathcal{R}^{il} = \epsilon^{ijk} \epsilon^{lmn} \mathcal{R}_{jmn}$ in the 4-dimensional background in the closed superstring theory as discussed above. When this field is in the $\mu = 3$ direction the deformation is proportional to $(J^3 + \psi^1 \psi^2) \bar{\mathcal{J}}$ and the right moving current $\bar{\mathcal{J}}$ is normalized as $\langle \bar{\mathcal{J}}(1) \bar{\mathcal{J}}(0) \rangle = k_g/2$. Rewriting the currents in Euler angles, i.e. $J^3 = k(\partial\gamma + \cos\beta\partial\alpha)$,

$\overline{J}^3 = k(\overline{\partial}\alpha + \cos\beta\overline{\partial}\gamma)$, we get for the perturbation of the (heterotic) action in (2), the following expression:

$$\delta S_4 = \frac{\sqrt{k k_g} H}{2\pi} \int d^2 z (\partial\gamma + \cos\beta\partial\alpha) \overline{J}. \quad (7)$$

$$S_4 + \delta S_4 = \frac{k}{4} I_{SO(3)}(\alpha, \beta, \gamma) + \delta S_4 + \frac{k_g}{4\pi} \int d^2 z \partial\phi \overline{\partial}\phi = \frac{k}{4} I_{SO(3)}(\alpha, \beta, \gamma + 2\sqrt{\frac{k_g}{k}} H\phi) + \frac{k_g(1-2H^2)}{4\pi} \int d^2 z \partial\phi \overline{\partial}\phi.$$

This shows the marginal character of the magnetic deformation where we have chosen for the currents J and \overline{J} their bosonizations $\partial\phi$ and $\overline{\partial}\phi$ correspondingly.

To the perturbation (7) there is a corresponding background determined by background fields like a graviton $G_{\mu\nu}$, gauge fields $F_{\mu\nu}^a$, an antisymmetric field (three form) $H_{\mu\nu\rho}$ and a dilaton Φ , which, in turn, are solutions to the following equations of motion:

$$\begin{aligned} \frac{3}{2} \left[4(\nabla\Phi)^2 - \frac{10}{3}\square\Phi - \frac{2}{3}R + \frac{1}{12g^2} F_{\mu\nu}^a F^{a,\mu\nu} \right] &= C \\ R_{\mu\nu} - \frac{1}{4}H_{\mu\nu}^2 - \frac{1}{2g^2} F_{\mu\rho}^a F_{\nu}^{a\rho} + 2\nabla_\mu \nabla_\nu \Phi &= 0 \\ \nabla^\mu [e^{-2\Phi} H_{\mu\nu\rho}] &= 0 \\ \nabla^\nu [e^{-2\Phi} F_{\mu\nu}^a] - \frac{1}{2} F^{a,\nu\rho} H_{\mu\nu\rho} e^{-2\Phi} &= 0 \end{aligned} \quad (8)$$

These equations can be derived by the variations of the following effective 4-dimensional gauge theory action:

$$S = \int d^4 x \sqrt{G} e^{-2\Phi} \left[R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 - \frac{1}{4g^2} F_{\mu\nu}^a F^{a,\mu\nu} + \frac{C}{3} \right] \quad (9)$$

where C is the l.h.s. of the first equation in (8). Here we set $g_{str} = 1$ and for the gauge coupling $g^2 = 2/k_g$. The fields $F_{\mu\nu}^a, H_{\mu\nu\rho}$ are usually defined by $F_{\mu\nu}^a = \partial_\mu A_\nu - \partial_\nu A_\mu + f^{abc} A_\mu^b A_\nu^c$, $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2g^2} [A_\mu^a F_{\nu\rho}^a - \frac{1}{3} f^{abc} A_\mu^a A_\nu^b A_\rho^c] + \text{permutations}$. The constants f^{abc} are structure constants of the gauge group and A_μ^a is the effective gauge field. One can notice that the term in $H_{\mu\nu\rho}$ enclosed by square brackets is the Chern-Simons term for the gauge potential A_μ^a . Now a solution of these equations for the background agreeing with the deformation (7), reads:

$$\begin{aligned} G_{00} &= 1, \quad G_{\beta\beta} = \frac{k}{4}, \quad G_{\alpha\gamma} = \frac{k}{4}(1-2H^2)\cos\beta \\ G_{\alpha\alpha} &= \frac{k}{4}(1-2H^2\cos^2\beta), \quad G_{\gamma\gamma} = \frac{k}{4}(1-2H^2), \\ B_{\alpha\gamma} &= \frac{k}{4}\cos\beta \\ A_\alpha &= g\sqrt{k}H\cos\beta, \quad A_\gamma = g\sqrt{k}H, \quad \Phi = \frac{x^0}{\sqrt{k+2}}. \end{aligned} \quad (10)$$

where H is the magnetic field as in (7).

Similarly, when gravitational marginal deformations like in the vertex $V_{gr} = \mathcal{R}(J^3 + \psi^1\psi^2)\overline{J}^3$ are included one can derive a corresponding exact background of string

The new σ -model with action $S_4 + \delta S_4$ is again conformally invariant for all orders in α' since:

theory by σ -model calculations [16, 18]. Again, one obtains for the fields in this background by solving the effective field theory equations (8)[18]:

$$\begin{aligned} G_{00} &= 1, \quad G_{\beta\beta} = \frac{k}{4} \\ G_{\alpha\alpha} &= \frac{k}{4} \frac{(\lambda^2+1)^2 - (8H^2\lambda^2 + (\lambda^2-1)^2)\cos^2\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2} \\ G_{\gamma\gamma} &= \frac{k}{4} \frac{(\lambda^2+1)^2 - (8H^2\lambda^2 + (\lambda^2-1)^2)\cos^2\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2} \\ G_{\alpha\gamma} &= \frac{k}{4} \frac{4\lambda^2(1-2H^2)\cos\beta + (\lambda^4-1)\sin^2\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2} \\ B_{\alpha\gamma} &= \frac{k}{4} \frac{\lambda^2-1+(\lambda^2+1)\cos\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2} \\ A_\alpha &= 2g\sqrt{k} \frac{H\lambda\cos\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2} \\ A_\gamma &= 2g\sqrt{k} \frac{H\lambda}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2} \\ \Phi &= \frac{t}{\sqrt{k+2}} - \frac{1}{2} \log \left[\lambda + \frac{1}{\lambda} + (\lambda - \frac{1}{\lambda}) \cos\beta \right] \end{aligned} \quad (11)$$

We remark that the dependence on λ (see App. A) shows the existence of gravitational backreaction which was absent in the purely magnetic deformed background (10).

III.2. The mass gap in the background with linear dilaton

In field theory a dilaton Φ couples to a massless bosonic field T in a universal fashion:

$$S[\Phi, T] = \int e^{-2\Phi} \partial_M T \partial^M T. \quad (12)$$

which can be rewritten by introducing a new field $U = e^{-\Phi} T$ to:

$$S[\Phi, U] = \int \partial_M U \partial^M U + [\partial^2\Phi - \partial_M \Phi \partial^M \Phi] U \quad (13)$$

Thus for a linear dilaton $\Phi = q_M X^M$ the field U becomes massive with mass square $M^2 = q_M q^M$ (for X^M space-like). In this way the massless boson T is mapped to the boson U with the mass M . However this mechanism does not work in case of massless free fermions. But in a four dimensional spacetime, the chiral fermion ψ can be coupled to an antisymmetric tensor $H_{\mu\nu\rho}$ like:

$$S[\psi, H] = \int \overline{\psi} \gamma^\mu \left[\overleftrightarrow{\partial}_\mu + H_\mu \right] \psi$$

where $H_\mu = \varepsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}$ is the dual of the antisymmetric tensor $H_{\nu\rho\sigma}$. This system can be embedded into a string background with the fields: Φ and H_{MNP} . Then by using one-loop string equations:

$$\begin{aligned} R_{MN} &= -2\nabla_M \nabla_N \Phi + \frac{1}{4} H_{MPR} H_N{}^{PR}, \\ \nabla_L (e^{-2\Phi} H_{MN}^L) &= 0, \\ \nabla^2 \Phi - 2(\nabla \Phi)^2 &= -\frac{1}{12} H^2, \end{aligned} \quad (14)$$

one gets for the linear dilaton $\Phi = q_M X^M$ the following relation:

$$q_M q^M = \frac{1}{6} H^2$$

and the scalar curvature R is:

$$R = \frac{3}{2} q_M q^M.$$

If non-vanishing components of q_M , X^M and H_{MNP} are in the direction of the four dimensional space one obtains:

$$q^\mu \sim \varepsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}.$$

Thus the Dirac operator acquires a mass gap proportional to $q_\mu q^\mu$.

The problem to embed a four dimensional fermion system in the exact string background was considered in [20].

Now we consider the case when the four dimensional space is represented by the $\mathbb{R}_\phi \times SU(2)_k$ part of the string background. Then the linear dilaton is given by $\Phi = QX^0$ where Q is related to the level k of the WZW model on $SU(2)$ by $Q = (k+2)^{-1/2}$. Therefore the corresponding CFT has the same central charge like in the flat case. Hence the direct consequence of (13) is that the massless bosons acquire the mass gap $\Delta M^2 = \mu^2 = (k+2)^{-1}$.

Given the exact string background with four dimensional part $\mathbb{R}_\phi \times SU(2)_k$, we consider the perturbation of this theory by gauge and gravitational fields. Now we will analyze the existence of the mass gap by some explicit calculations in the perturbed theory.

The minimal value of the mass of bosonic states in the deformed theory reads:

$$\begin{aligned} M_{\min}^2(Q, I) &= \frac{1}{2} (Q^2 - 1) + \frac{1}{k+2} \left[(|I| + 1/2)^2 - (Q + I)^2 \right] + \\ &\quad \frac{1}{2} (1 + \sqrt{1 + F^2}) \left(\frac{Q+I}{\sqrt{k+2}} + eH \right)^2, \end{aligned} \quad (15)$$

where $|Q| = 1$ and $|I| = 0, 1, \dots, k/2$ and Q, I, P are the zero-modes of the corresponding currents in this theory (see [18] and Appendix A). We obtain for the magnetic field $H = F / (\sqrt{2} (1 + \sqrt{1 + F^2}))$ and the electric charge $e = \sqrt{2P} / \sqrt{k_g}$. In the discussed case the function F is given by:

$$F = \sinh(2x),$$

where x is a parameter of the group $O(1, 1)$. Thus the magnetic field H takes the form:

$$H = \frac{1}{\sqrt{2}} \tanh x.$$

For $x = 0$ the minimal value of the mass is:

$$M_{\min}^2(Q, I) = \frac{1}{2} (Q^2 - 1) + \frac{(|I| + 1/2)^2}{k+2}.$$

Since $|Q| = 1$ the first term vanishes and one gets:

$$M_{\min}^2(Q, I) = \frac{(|I| + 1/2)^2}{k+2}.$$

This value is proportional to the mass gap obtained from the linear dilaton.

Now let us compute the following difference of mass squares: $\Delta M_{(1)}^2 = M_{\min}^2(+1, I) - M_{\min}^2(-1, I)$:

$$\Delta M_{(1)}^2 = \frac{4}{k+2} S(I),$$

where the function $S(I)$ is equal to:

$$S(I) = 2I \left(-1 + \sqrt{1 + F^2} \right) + \frac{\sqrt{2}}{4} eF \sqrt{k+2}.$$

$$\Delta M_{(1)}^2 = \frac{16I \sinh^2 x}{k+2} + \frac{\sqrt{2}}{4} \frac{e \sinh(2x)}{\sqrt{k+2}}.$$

For $x = 0$ (the magnetic perturbation is switched off at this stage of the theory) one gets: $\Delta M_{(1)}^2 = 0$, hence the mass is minimal for any $Q = \pm 1$. Next we consider another difference of square masses of the string states: $\Delta M_{(2)}^2 = M_{\min}^2(Q, I+1) - M_{\min}^2(Q, I)$. In this case we get:

$$\begin{aligned} \Delta M_{(2)}^2(I; x) &= \\ \frac{1}{k+2} [2I + |I+1| - |I| + 1 + (2Q + 2I + 1) \sinh^2 x] &+ \frac{e \sinh(2x)}{\sqrt{2} \sqrt{k+2}}. \end{aligned} \quad (16)$$

For $x = 0$ we get:

$$\Delta M_{(2)}^2(I; 0) = \frac{1}{k+2} [2I + |I+1| - |I| + 1].$$

Because:

$$2I + |I+1| - |I| + 1 = \begin{cases} 2I + 2 & \text{for } I \geq 0 \\ 4I + 2 & \text{for } I \in [-1, 0] \\ 2I & \text{for } I \leq -1 \end{cases}$$

we obtain:

$$\Delta M_{(2)}^2(I; 0) = \begin{cases} \frac{2I+2}{k+2} & \text{for } I \geq 0 \\ \frac{2I}{k+2} & \text{for } I \leq -1 \end{cases}$$

So again linear dilaton mass gap is smaller than this difference.

IV. THE ENERGY SPECTRA IN DIMENSION 4

IV.1. Field theory vs. string theory spectra of charged particles in standard 4-space

In case of the magnetic field on S^3 we choose for the vector potential A_μ in the model (10):

$$A_\alpha = H \cos \beta, \quad A_\beta = 0, \quad A_\gamma = H. \quad (17)$$

The Hamiltonian for a particle of electric charge e moving on S^3 is given by

$$\bar{\mathbf{H}} = \frac{1}{\sqrt{\det G}} (\partial_\mu - ieA_\mu) \sqrt{\det G} G^{\mu\nu} (\partial_\nu - ieA_\nu). \quad (18)$$

where we assume the standard metric $G_{\mu\nu}$ on S^3 . The energy spectrum for $\bar{\mathbf{H}}$ is then given by:

$$\Delta E_{j,m} = \frac{1}{R^2} [j(j+1) - m^2 + (eH - m)^2] \quad (19)$$

where $j \in \mathbb{Z}$ and $-j \leq m \leq j$ (like in the case of the group $SO(3)$). In the flat limit we retrieve the Landau spectrum of spinless particles in the 3-dimensional space :

$$\Delta E_{n,p_3} = e\tilde{H}(2n+1) + p_3^2 + \mathcal{O}(R^{-1}) \quad (20)$$

where we rescale eH to $e\tilde{H} = e\tilde{H} + \kappa R + \mathcal{O}(1)$ and $m = e\tilde{H}R^2 + (p_3 + \kappa) + \mathcal{O}(1)$. This behavior can be derived by rewriting the spectrum (19) by using the new parameter n : $j = |m| + n$ for $|m|, n \in \mathbb{N}$ into $\Delta E_{n,m} = \frac{1}{R^2} [n(n+1) + |m|(2n+1)] + \left(\frac{eH-m}{R}\right)^2$.

Let us, again following [18], calculate the spectrum in case of the exact string background (10). If one choose the metric (10) of the background then one is able to derive the eigenvalues of the Hamiltonian (19) with the result [18]:

$$\Delta E_{j,m} = \frac{1}{R^2} \left[j(j+1) - m^2 + \frac{(eHR - m)^2}{(1 - 2H^2)} \right]. \quad (21)$$

With the abbreviations $n \in \mathbb{N}$ by $j = |m| + n$, $|m| = 0, 1/2, 1, \dots$ we can rewrite the spectrum (21) as:

$$\Delta E_{n,m} = \frac{1}{R^2} [n(n+1) + |m|(2n+1)] + \left(\frac{eHR - m}{R\sqrt{1 - 2H^2}} \right)^2 \quad (22)$$

where the energy spectrum contains the corrections due to the H field appearing in the exact string background (10). But we remark that the Hamiltonian (18) is the Hamiltonian of a 4-dimensional theory.

Using [17, 18] and the appendix A, one can also calculate the energy spectrum in this background in comparison to (22). Importantly, we are able to get from one spectrum to the other by the following rules:

$$\begin{aligned} R^2 &\rightarrow k+2, \quad m \rightarrow \mathcal{Q} + J^3, \quad e \rightarrow \sqrt{\frac{2}{k_g}} \bar{\mathcal{Q}} \\ H &\rightarrow \frac{F}{\sqrt{2(1+\sqrt{1+F^2})}} = \frac{1}{2\sqrt{2}} \left[F - \frac{F^3}{4} + \mathcal{O}(F^5) \right] \end{aligned} \quad (23)$$

where

$$F^2 = \langle F_{\mu\nu}^a F_a^{\mu\nu} \rangle = \int_{SU(2)} F_{\mu\nu}^a F_a^{\mu\nu} d\text{vol}(SU(2))$$

is the integrated (squared) field strength with $H_i^a = \epsilon^{ijk} F_{jk}^a$.

For a particle with spin S (setting $S = Q$) we obtain the following modifications of the spectrum due to the above rules[18]:

$$\Delta E_{j,m,S} = \frac{1}{k+2} \left[j(j+1) - (m+S)^2 + \frac{(eHR - m - S)^2}{(1 - 2H^2)} \right]. \quad (24)$$

The next step is the inclusion of the gravitational back-reactions. One starts with the string background (11) and computes again the eigenvalues of (18). The result for scalar particles is [18]:

$$\Delta E_{j,m,\bar{m}} = \frac{1}{R^2} \left[j(j+1) - m^2 + \frac{(2ReH - (\lambda + \frac{1}{\lambda})m - (\lambda - \frac{1}{\lambda})\bar{m})^2}{4(1 - 2H^2)} \right] \quad (25)$$

where $-j \leq m, \bar{m} \leq j$. Again, we obtain for even k (see (A7) in the Appendix A) the corresponding rules in the case where gravity backreactions are included:

$$\begin{aligned} R^2 &\rightarrow k+2, \quad m \rightarrow \mathcal{Q} + J^3, \quad e \rightarrow \sqrt{\frac{2}{k_g}} \bar{\mathcal{P}}, \quad \bar{m} \rightarrow \bar{J}^3 \\ H^2 &\rightarrow \frac{1}{2} \frac{F^2}{F^2 + 2(1 + \sqrt{1 + F^2 + \mathcal{R}^2})}, \quad \lambda^2 = \frac{1 + \sqrt{1 + F^2 + \mathcal{R}^2} + \mathcal{R}}{1 + \sqrt{1 + F^2 + \mathcal{R}^2} - \mathcal{R}} \end{aligned} \quad (26)$$

where

$$\mathcal{R}^2 = \langle R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \rangle = \int_{SU(2)} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} d\text{vol}(SU(2))$$

is the integrated squared scalar curvature and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the „squashed” $SU(2) = S^3$ in

the deformed background.

IV.2. Exotic 4-geometry limit

In the previous sections we obtained a dictionary (26) to modify the spectrum (25) of a scalar particle with charge e by the influence of a magnetic field H and the inclusion of the gravitational backreaction. Then the dependence on the even level k emerges:

$$\Delta E_{j,m,\overline{m}}^k = \frac{1}{k+2} [j(j+1) - m^2] + \frac{(2\sqrt{k+2}eH - (\lambda + \frac{1}{\lambda})m - (\lambda - \frac{1}{\lambda})\sqrt{(1+2/k)\overline{m}})^2}{4(k+2)(1-2H^2)}. \quad (27)$$

Based on our discussion above and in Sec. II we consider (27) as the modification of the spectrum of a scalar quantum particle with charge when moving through a 4-region with exotic geometry \mathbb{R}_k^4 where the scaling of H is required on the exotic background. Even though we do not determine explicitly this scaling (this would require the unknown exotic functions on \mathbb{R}_k^4) the spectrum is exactly derived. Moreover, the local description of an exotic 4-space remains similar to the standard case.

Now we want to comment on the possible physics behind the emergence of these quantum gravitational effects induced by an exotic \mathbb{R}_k^4 in spacetime. Each exotic \mathbb{R}_k^4 , $k = 1, 2, \dots$ is a member of the radial family of exotic \mathbb{R}^4 in the standard \mathbb{R}^4 . This means that exotic smoothness of \mathbb{R}_k^4 is confined to the open subset of the radii $\sim \sqrt{k}$ [5]. However this exotic smooth 4-region cannot be extended smoothly (with respect to the standard coordinates) over bigger area and larger radii or cannot be smoothly glued into the bigger standard \mathbb{R}^4 . Let us consider the exotic structure of \mathbb{R}_k^4 in the „infinite” extendible \mathbb{R}^4 denoted by $\check{\mathbb{R}}_k^4 \subset \mathbb{R}^4$. From the point of view of standard smoothness on \mathbb{R}^4 , the exotic smoothness structure of $\check{\mathbb{R}}_k^4$ can be interpreted as a matter sources for gravity. Let us suppose that there is a definite density of gravitational energy in this $\check{\mathbb{R}}_k^4$. Next let us shrink this $\check{\mathbb{R}}_k^4$ to the region of a finite diameter $\sim h_d(\sqrt{k})$ in the standard 4-space. [?] The amount of energy is contracted and confined in a smaller bounded 4-region and for some, appropriately small diameter, quantum gravity effects become important and dominate. In the same time every exotic \mathbb{R}_k^4 determines the WZ term of the $SU(2)_k$ WZW model and the geometry of the end determines the $SU(2)_k \times \mathbb{R}$ via its smoothness structure (see Sec. II and [5]). When a particle moves through such exotic, contracted, 4-region the QG effects dominates and are calculated via $SU(2)_k$ WZW model as in string theory on these backgrounds.

When the radii \sqrt{k} of the $SU(2)$ (seen as 3-sphere) is assigned to the size of the 4-region in spacetime, one has the dominance of the classical 4-geometry for large radii. A particle moves via exotic smooth trajectories determined with respect to the Riemannian geometry of

the non-contracted \mathbb{R}_k^4 . The QG effects are calculated via models of 2-d CFT hence they are preserved under the 2-d scaling of coordinates (conformal transformations). This means that the non-contracted $\check{\mathbb{R}}_k^4$ contains also some QG ingredients but these are dominated by classical Riemannian 4-geometry and are negligible.

Let us consider the change of the smooth structures \mathbb{R}_k^4 with the size of the region \sqrt{k} in \mathbb{R}^4 , then the non-contracted limit corresponds to the flat \mathbb{R}^4 and this is precisely the limit of large k of $SU(2)_k$ WZW. The classical limit of 4-d QG is thus achieved when $k \rightarrow \infty$ and in this limit, in the case of Eq. (24) (without gravitational backreactions from H), there appears the continuum spectrum. Thus when we shift the standard 4-smoothness of the \mathbb{R}^4 into an exotic \mathbb{R}_k^4 and confine it into the 4-region of diameter $\sim \sqrt{k}$, the energy spectrum of a charged particle moving through the exotic region, without inclusion of gravitational backreactions from the magnetic field, becomes quantized. Moreover, the mass gap in the matter spectrum appears, $\mu \sim \frac{1}{\sqrt{k+2}}$, which disappears in the flat \mathbb{R}^4 limit (Sec. III.2). In the „classical” gravity (flat) limit $k \rightarrow \infty$, the spectrum (27) reflects some modified Landau levels including gravitational backreactions. This spectrum is further modified as in (27) by the presence of an exotic \mathbb{R}_k^4 . Again, the mass gap appears due to the exotic 4-smoothness underlying the 4D spacetime.

From the interpretation above we got a direct indication of quantum gravitational effects derived from the non-standard smooth metrics on \mathbb{R}_k^4 . More sophisticated connections with QFT and quantization were worked out in [7] containing deep topological and differential-geometric results.

The scaled field H is only *locally* constant on the exotic \mathbb{R}_k^4 . Then gravitational backreactions from the field H are derived as the quantum response of the exotic background \mathbb{R}_k^4 due to its Riemannian curvature, i.e. the density of gravitational energy. On the contracted exotic \mathbb{R}_k^4 , however, there dominates the quantum gravity component which deform consistently the spectra of quantum particles. Taking the exotic \mathbb{R}_k^4 limit (finite k) instead of flat \mathbb{R}^4 ($k \rightarrow \infty$) explains the dependence on k . Fixing k

in (27) refers to the amount of gravity which is contained in the geometry of the \mathbb{R}_k^4 , the quantum numbers j , m , \bar{m} are related with symmetries of the (contracted) exotic \mathbb{R}_k^4 , the gravitational backreactions of H are encoded in the dependence on the modulus λ .

However, this kind of overlapping between *geometry*, *quantum matter* and *gravity* predicts in fact a new way in which QG is rooted in 4-dimensions and how it interacts with matter and geometry. Namely, let us recall that GR predicts the link between the geometry of spacetime and matter-energy fields by the Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ where $G_{\mu\nu}$ is now the Einstein tensor and $T_{\mu\nu}$ the energy-momentum tensor. Quantum fields carry energy and mass i.e. according to GR they disturb the curvature of spacetime but the geometry affects the propagation of quantum matter fields. In the presented paper, we have calculated the influence of the smooth (exotic) 4-geometry on the quantum particles and field H in the 4-dimensional spacetime. The calculations went through the quantum level of gravity as in 10d superstring theory and dominated the classical smooth Riemannian 4-geometry. This scenario shows that quantizing the gravity field and quantum matter in dimension 4 can be exactly solved via non-standard smoothness on \mathbb{R}^4 , where the necessary tools are 10 dimensional. The particular case presented in the paper has a more universal meaning: it uncovers a new way of interaction of gravity and quantum matter (see also [9]). We will further work on this important issue (cf. [2, 9, 21, 22]).

V. SUMMARY

In this paper we presented the influence of the exotic \mathbb{R}_k^4 on some energy spectra of charged particles in spacetime whose smoothness is defined with respect to this exotic \mathbb{R}_k^4 . In our model, the effects are exactly calculable via techniques of superstring theory in curved backgrounds and 2d CFT. The whole procedure shows that the Riemannian geometry of exotic \mathbb{R}_k^4 emerges in the correct classical limit for the quantum gravity represented by superstring theory at least in the case of the $SU(2)_k \times \mathbb{R}_\phi \times K^6$ backgrounds. Why is superstring theory well suited for exotic \mathbb{R}_k^4 effects in 4-dimensions? As the proposed model shows superstring theory is a universal and consistent way to describe the change of the 4-dimensional part of the background from the flat \mathbb{R}^4 to the curved $SU(2)_k \times \mathbb{R}_\phi$. The supersymmetric theory on these backgrounds is minimally perturbed compared to the flat one [18], and the exact results, such as QG corrections, are derivable. Exactly this change of 4-geometry, the shift from the flat 4-space to the exotic \mathbb{R}_k^4 was derived from topological considerations. Thus we have a way to deal with the quantum effects of gravity which is confined to exotic \mathbb{R}_k^4 . Superstring theory is UV finite and unifies gravity with other interactions which takes place in large energies $\mathcal{O}(M_{str}) \simeq 10^{17}$ GeV. However, the excited string states become important for these energies and effective string field theory has to be

modified. The results in the paper show that a modified 4-dimensional field theory on some exotic \mathbb{R}_k^4 should be considered in this regime.

Hence, this \mathbb{R}_k^4 Riemannian geometry becomes an important ingredient of the classical limit for superstring theory so that 4-dimensional field theory emerging in the classical limit with well defined QG corrections should be formulated on spacetimes with exotic 4-geometry (cf. [7]). The appearance of the mass gap in the matter spectrum due to nonstandard smoothness is an indication in favor of this point of view. Moreover the curvature of the string background considered here (hence the mass gap) can act as the infrared regulator for string and field theory (see e.g. [19]), so exotic \mathbb{R}_k^4 can play a role as a natural 4-geometric regulator. These interesting points we will address in our forthcoming work.

The results in the paper show that gravity (classical and quantum), geometry and quantum fields can be related with each other in a different way than usually prescribed by QFT on semi-classical backgrounds or by various techniques from superstring theory. The fact that superstring theory is UV finite and is the theory of quantum gravity containing other interactions is crucial for the results. The entire approach works due to the existence of non-standard geometries exclusively in dimension 4 which appear as new and important ingredients of a (final) theory of 4-dimensional QG.

Appendix A: The deformed exact string spectra

Following [17, 18] we present now the CFT calculations leading to the description of the string backgrounds deformed by magnetic and corresponding gravitational marginal deformations. In the case of a single magnetic field F the operators corresponding to truly marginal deformations and in the case of the current-current interactions, are given by the bilinear product of currents (6): $V_F = F \frac{(J^3 + \psi^1 \psi^2)}{\sqrt{k+2}} \frac{\bar{J}}{\sqrt{k_g}}$ where J^3 , \bar{J}^3 are the $SU(2)$ currents, J , \bar{J} are holomorphic and antiholomorphic ones, and the right moving current \bar{J} is normalized as $\langle \bar{J}(1)\bar{J}(0) \rangle = k_g/2$. The corresponding gravitational deformation reads: $V_{gr} = \mathcal{R} \frac{(J^3 + \psi^1 \psi^2) \bar{J}^3}{\sqrt{k+2}\sqrt{k}}$.

Let us include these marginal deformations V_F and V_{gr} as $O(1,1)$ boost in the lattice of charges of the theory. The effects will be encoded in the zero-modes of the $SU(2)_k$ currents, J^3 , \bar{J}^3 , i.e. I , \bar{I} , the zero-modes of the holo- (antiholo-)morphic currents, J , \bar{J} , i.e. \mathcal{P} , $\bar{\mathcal{P}}$, and the zero-mode of the holomorphic helicity current, $\psi^1 \psi^2$, which is denoted by \mathcal{Q} . Then the zero-modes of the algebra are:

$$L_0 = \frac{\mathcal{Q}^2}{2} + \frac{I^2}{2} + \dots, \quad \bar{L}_0 = \frac{\bar{\mathcal{P}}^2}{k_g} + \dots$$

which gives rise to the relevant for the V_F perturbation part:

$$L_0 = \frac{(\mathcal{Q} + I)^2}{k+2} + \frac{k}{2(k+2)} \left(\mathcal{Q} - \frac{2}{k} I \right)^2 + \dots \quad (\text{A1})$$

The $O(1, 1)$ boost mixes the holomorphic zero-mode current $I + \mathcal{Q}$ with the antiholomorphic $\bar{\mathcal{P}}$:

$$\begin{aligned} L'_0 &= \left(\cosh x \frac{\mathcal{Q}+I}{\sqrt{k+2}} + \sinh x \frac{\bar{\mathcal{P}}}{\sqrt{k_g}} \right)^2 + \frac{k}{2(k+2)} \left(\mathcal{Q} - \frac{2}{k} I \right)^2 + \dots \\ \bar{L}'_0 &= \left(\sinh x \frac{\mathcal{Q}+I}{\sqrt{k+2}} + \cosh x \frac{\bar{\mathcal{P}}}{\sqrt{k_g}} \right)^2 + \dots \end{aligned} \quad (\text{A2})$$

$$\delta L_0 = \left[\frac{\mathcal{R}\bar{I}}{\sqrt{k}} + \frac{F^2}{\sqrt{k_g}} \right] \frac{\mathcal{Q}+I}{\sqrt{k+2}} + \left(\sqrt{1+\mathcal{R}^2+F^2} - 1 \right) \left[\frac{(\mathcal{Q}+I)^2}{2(k+2)} + \frac{1}{2(\mathcal{R}^2+F^2)} \left(\frac{\mathcal{R}\bar{I}}{\sqrt{k}} + \frac{F\bar{\mathcal{P}}}{\sqrt{k_g}} \right)^2 \right]. \quad (\text{A3})$$

This perturbation gives the mass spectra $L_0 = M_L^2$ ($\bar{L}_0 = M_R^2$):

$$\begin{aligned} M_L^2 &= -\frac{1}{2} + \frac{\mathcal{Q}^2}{2} + \frac{1}{2} \sum_{i=1}^3 \mathcal{Q}_i^2 + \frac{(j+1/2)^2 - (\mathcal{Q}+I)^2}{k+2} + E_0 + \\ &\quad + \frac{1+\sqrt{1+F^2}}{2} \left[\frac{\mathcal{Q}+I}{\sqrt{k+2}} + \frac{F\bar{\mathcal{P}}}{\sqrt{k_g}(1+\sqrt{1+F^2})} \right]^2 + \\ &\quad + \frac{1+\sqrt{1+\mathcal{R}^2}}{2} \left[\frac{\mathcal{Q}+I}{\sqrt{k+2}} + \frac{\mathcal{R}\bar{I}}{\sqrt{k}(1+\sqrt{1+\mathcal{R}^2})} \right]^2 \end{aligned} \quad (\text{A4})$$

and with the help of the following gravitational backreaction moduli:

$$\lambda = \sqrt{\mathcal{R} + \sqrt{1+\mathcal{R}^2}}, \frac{1}{\lambda} = \sqrt{-\mathcal{R} + \sqrt{1+\mathcal{R}^2}} \quad (\text{A5})$$

the spectra can be obtained:

$$\begin{aligned} M_L^2 &= -\frac{1}{2} + \frac{\mathcal{Q}^2}{2} + \frac{1}{2} \sum_{i=1}^3 \mathcal{Q}_i^2 + \frac{(j+1/2)^2 - (\mathcal{Q}+I)^2}{k+2} + E_0 + \\ &\quad + \frac{1+\sqrt{1+F^2}}{2} \left[\frac{\mathcal{Q}+I}{\sqrt{k+2}} + \frac{F\bar{\mathcal{P}}}{\sqrt{k_g}(1+\sqrt{1+F^2})} \right]^2 + \\ &\quad + \frac{1}{4} \left[\left(\lambda + \frac{1}{\lambda} \right) \frac{\mathcal{Q}+I}{\sqrt{k+2}} + \left(\lambda + \frac{1}{\lambda} \right) \frac{\bar{I}}{\sqrt{k}} \right]^2. \end{aligned} \quad (\text{A6})$$

The middle lines in (A4) and (A6) express the effect of the single constant magnetic field while the last lines - the gravitational backreactions. \mathcal{Q}_i , $i = 1, 2, 3$ refer to the helicity operators corresponding to the internal left fermions in the background $SU(2)_k \times \mathbb{R}_\phi \times K^6$ and $j = 0, 1, 2, \dots, p$ for $k = 2p$.

Next we include the V_{gr} deformation. This deformation is symmetric hence we perform in addition to V_F the $O(2)$ transformation which mixes the antiholomorphic \bar{I} , $\bar{\mathcal{P}}$. After substituting $F = \sinh(2x)$ we have the following perturbation $\delta L_0 = L'_0 - L_0$ in L_0 (and \bar{L}_0):

Similarly, the right moving mass spectra read:

$$\begin{aligned} M_R^2 &= -1 + \frac{P^2}{k_g} + \frac{(j+1/2)^2 - (\mathcal{Q}+I)^2}{k+2} + \bar{E}_0 + \\ &\quad + \frac{1+\sqrt{1+F^2}}{2} \left[\frac{\mathcal{Q}+I}{\sqrt{k+2}} + \frac{F\bar{\mathcal{P}}}{\sqrt{k_g}(1+\sqrt{1+F^2})} \right]^2 + \\ &\quad + \frac{1}{4} \left[\left(\lambda + \frac{1}{\lambda} \right) \frac{\mathcal{Q}+I}{\sqrt{k+2}} + \left(\lambda + \frac{1}{\lambda} \right) \frac{\bar{I}}{\sqrt{k}} \right]^2. \end{aligned} \quad (\text{A7})$$

For the completeness let us write the partition function $Z^W(\tau, \bar{\tau})$ for the curved 4-spacetime $W = SU(2)_{k=2p} \times \mathbb{R}_Q$ in terms of flat 4-space function Z_0 (still for $k = 2p$):

$$Z^W(\tau, \bar{\tau}) = \text{Im}\tau^{3/2} |\eta(\tau)|^6 \frac{\sum_{\gamma, \delta=0}^1 Z_{so(3)} \left[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix} \right]}{2V} Z_0(\tau, \bar{\tau}) \quad (\text{A8})$$

and $Z_{so(3)} \left[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix} \right] = e^{-i\pi\gamma\delta k/2} \sum_{l=0}^k e^{i\pi\delta l} \chi_l \chi_{(1-2\gamma)l+\gamma k}$ where $\chi_l(\tau)$ are the $SU(2)_k$ characters:

$$\chi_l(\tau) = \frac{\vartheta_{l+1,k+2}(\tau, v) - \vartheta_{-l-1,k+2}(\tau, v)}{\vartheta_{1,2}(\tau, v) - \vartheta_{-1,2}(\tau, v)} \Big|_{v=0} \quad (\text{A9})$$

with the level k ϑ -functions.

Together with the mass spectra (A6), (A7) we have the complete and exact string spectra in the presence of the constant magnetic field and curvature \mathcal{R} . Then, based on the mass spectra and (25) one derives the dictionary rules as in (26) [18], where $\lambda^2 \approx 1 + \mathcal{R} + \mathcal{O}(F^2, \mathcal{R}^2)$ which agrees with (A5) up to $\mathcal{O}(F^2)$.

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